SOLUTIONS OF EXERCISES.

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A homogeneous heavy rod is hung from a fixed point by elastic threads of given length fastened at its extremities. Find the position of equilibrium.

[*W. M. Thornton.*]

SOLUTION.

Assuming the rod to be of uniform section, the position of equilibrium is the same as if W, the weight of the rod, was concentrated at its middle point, through which passes the vertical through the point of suspension.

Let l and l' be the given lengths of the strings, and μ, μ' such weights as would stretch them to double their original lengths. Let dl and dl' be their respective elongations, and θ the angle between the strings in the position of equilibrium.

T and T' being the tensions in the strings, we have from statical relations

$$T^2 + T'^2 + 2TT' \cos\theta = W^2; \tag{1}$$

also
$$T/T' = (l + dl)/(l + dl').$$

By Hooke's law,

$$T = \mu dl/l; \qquad T' = \mu' dl'/l'. \tag{2}$$

Hence

$$T/T' = \mu l' dl/\mu' l dl' = (l + dl)/(l' + dl'). \tag{3}$$

From (1) and (2) result

$$\left(\frac{\mu l' dl}{\mu' l dl'}\right)^2 + 1 + 2\,\frac{\mu l dl'}{\mu' l' dl}\cos\theta = \left(\frac{Wl'}{\mu' dl'}\right)^2;$$

and from the geometry of the figure we get

$$\left(\frac{l + dl}{l' + dl'}\right)^2 + 1 + 2\,\frac{l + dl}{l' + dl'}\cos\theta = \left(\frac{2x}{l' + dl'}\right)^2,$$

where x is the distance of the middle of the rod below the point of suspension.

These two equations give, by virtue of relation (3),

$$dl' = \frac{Wl'^2}{2\mu' x - Wl'}, \qquad \text{and also} \qquad dl = \frac{Wl^2}{2\mu x - Wl}.$$

Therefore the stretched lengths of the strings are

$$\frac{l'}{1 - \dfrac{Wl'}{2\mu' x}} \quad \text{and} \quad \frac{l}{1 - \dfrac{Wl}{2\mu x}},$$

where x is to be found from the geometrical relation

$$\left(\frac{l'}{1 - \dfrac{Wl'}{2\mu' x}}\right)^2 + \left(\frac{l}{1 - \dfrac{Wl}{2\mu x}}\right)^2 = 2(x^2 + a^2),$$

in which a is half the length of the given rod. This equation for x is of the sixth degree, which is best solved when μ and μ' are large, and therefore the elongations $Wl/2\mu$ and $Wl'/2\mu'$ are small compared with x, by using the median of the unstretched triangle as an approximate value of x and substituting successively in the first member of the equation. [*W. H. Echols.*]

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$f(x)$ and $\varphi(e^x)$ being rational algebraic functions of x and e^x, respectively, show that

$$\int f(x)\, \varphi(e^x)\, dx$$

depends upon integrals of three forms,

$$\int \frac{du}{\log u}, \quad \int \theta^m \tan\theta\, d\theta, \quad \int \frac{du}{(u+c)^r \log u}.$$

[*R. A. Harris.*]

SOLUTION.

There are, evidently, four forms to be considered; viz.,

$$\int x^m e^{ux} dx, \quad \int \frac{e^{ux}}{(x+a)^m}\, dx, \quad \int \frac{x^m}{(e^x+b)^n}\, dx, \quad \int \frac{dx}{(x+a)^m (e^x+b)^n}.$$

1. The first of these integrals can be found by successive integrations by parts.

2. The second, when treated in like manner, will be found to depend upon

$$\int \frac{e^{ux}}{x+a}\, dx.$$

This, in turn, depends upon the well-known transcendent

$$\int \frac{du}{\log u}.$$

3. Let us transform the expansion $\frac{x^m}{(e^x + b)^n} dx$ by letting $z = e^x$; then its integration may be made dependent upon integrals of the form

$$\int \frac{(\log z)^m}{2 + b} dz,$$

as follows from considering the partial fractions into which

$$\frac{(\log z)^m}{z(z + b)^n} dz \left(= \frac{x^m}{(e^x + b)^n} dx\right)$$

may be decomposed. Next, let $u = z/b$; then the above integral will depend upon

$$\int \frac{(\log u)^m}{u + 1} du.$$

If we let $\cos\theta + i \sin\theta = u^{\frac{1}{2}}$ it is not difficult to show that this integral depends upon

$$\int \theta^m \tan\theta \, d\theta.$$

When x and b are real and the latter is essentially negative, we should write $u = z/-b$, $\cos\theta - i \sin\theta = u^{\frac{1}{2}}$, thus making the integral in question depend upon

$$\int \theta^m \cot\theta \, d\theta,$$

which is, obviously, reducible to integrals of the form just written. It may be observed that when mod. θ lies between 0 and 1, mod. u lies between $1/e^2$ and e^2.

4. If

$$\frac{dx}{(x + a)^m (e^x + b)^u}$$

be integrated by parts $m - 1$ times, the integration will be seen to depend upon integrals of the form

$$\int \frac{e^{qx}}{(e^x + b)^r} \cdot \frac{dx}{x + a},$$

where $q \lesseqgtr m - 1$, $r \lesseqgtr m + u - 1$. This, in turn, depends upon

$$\int \frac{du}{(u + c)^r \log u}.$$

[*R. A. Harris.*]